

Finiteness properties of abc-equations $c = a + b$

Constantin M. Petridi
cpetridi@math.uoa.gr
cpetridi@hotmail.com

Abstract. We classify integer abc-equations $c = a + b$ (to be defined), according to their radical $R(abc)$ and prove that the resulting equivalence classes contain only a finite number of such equations. The proof depends on a 1933 theorem of Kurt Mahler.

1. abc-equations, their classes and types

For $i = 1, 2, \dots, \omega$, p_i, q_i, r_i denote primes, x_i, y_i, z_i integer variables ≥ 1 . $R(a) = \prod_{i=1}^{\omega} p_i$ is the radical of $a = \prod_{i=1}^{\omega} p_i^{x_i}$ and (a, b) the g.c.d. of a and b .

Integer equations $E : c = a + b$ are defined as abc-equations iff $1 \leq a < b$, $(a, b) = 1$. The radical $R(E)$ of E is defined by $R(E) = R(abc) = \prod_{p|abc} p = p_1 p_2 \cdots p_{\omega}$, where $p_1 < p_2 < \cdots < p_{\omega}$. Clearly $p_1 = 2$.

We embody all abc-equations with same radical $R(abc) = p_1 p_2 \cdots p_{\omega}$ into one class, denoted $C_{p_1 \cdots p_{\omega}}$. This induces a classification of the totality of all abc-equations into disjoint equivalence classes $C_{p_1 \cdots p_{\omega}}$, where a, b run independently over the abc -domain of constraint

$$1 \leq a < b, \quad (a, b) = 1.$$

To each abc-equation E belonging to the class $C_{p_1 \cdots p_{\omega}}$ we associate its type $T(E)$. To illustrate this point consider e.g. the equations $E_1 : p_1^{x_1} p_3^{x_2} = p_2^{x_3} + p_4^{x_4}$ and $E_2 : p_1^{y_1} = p_2^{y_2} p_4^{y_3} + p_3^{y_4}$ (assumed satisfiable in p_i, x_i, y_i , $i = 1, 2, 3, 4$). They both belong to the class $C_{p_1 p_2 p_3 p_4}$, but there is a formal difference between them. The first is of type $T(E_1) = (pp, p, p)$, the second of type $T(E_2) = (p, pp, p)$. For the first five cases, $\omega = 2, 3, 4, 5, 6$ the possible types are:

$\omega = 2$	$\omega = 3$	$\omega = 4$	$\omega = 5$	$\omega = 6$
$(p, 1, p)$	$(p, 1, pp)$	$(p, 1, ppp)$	$(p, 1, pppp)$	$(p, 1, ppppp)$
	(p, p, p)	(p, p, pp)	(p, p, ppp)	$(p, p, pppp)$
	$(pp, 1, p)$	$(pp, 1, pp)$	(p, pp, pp)	(p, pp, ppp)
		(pp, p, p)	$(pp, 1, ppp)$	$(pp, 1, pppp)$
		$(ppp, 1, p)$	(pp, p, pp)	(pp, p, ppp)
			$(ppp, 1, pp)$	(pp, pp, pp)
			(ppp, p, p)	$(ppp, 1, ppp)$
			$(pppp, 1, p)$	(ppp, p, pp)
				$(pppp, 1, pp)$
				$(pppp, p, p)$
				$(ppppp, 1, p)$

The type of an abc-equation $c = a + b$ may also be written as

$$T(E) = (\omega(c), \omega(a), \omega(b)),$$

where $\omega(n)$ is the number of different prime factors of n . Our notation puts more in evidence the combinatorial peculiarity of this notion.

For $\omega \geq 2$ the number of possible types is $\left\lceil \frac{(\omega+1)^2}{4} \right\rceil - 1$. We shall not give the proof (combinatorial) of this here, as for our present purpose it is only the finiteness of this number that is essential for a fixed ω .

2. Connection with the recurrence $P(n + \varphi(a)) = a + P(n)$

We refer and use notations and results of our article [2]. Since in the abc-equation $c = a + b$, c and b are coprime to a , they necessarily do appear in the sequence $P(n)$ with certain indexes. Denoting the index of b by n , it follows that the index of c is $n + \varphi(a)$. All abc-equations, therefore,

$$\begin{aligned} c &= a + b \\ 1 &\leq a < b \\ (a, b) &= 1, \end{aligned}$$

can also be written as

$$\begin{aligned} P(n + \varphi(a)) &= a + P(n) \\ 1 &\leq a \\ 1 &\leq n, \end{aligned}$$

where a and n run independently over the indicated domain.

The radical $R(abc)$ of the abc-equation takes the form

$$R(abc) = R\{aP(n)P(n + \varphi(a))\}.$$

3. Mahler's Theorem

In his 1933 paper [1] p.724-725, Kurt Mahler proved following theorem (freely translated from German), as a result of his investigations on the approximation of algebraic numbers.

Theorem (Mahler). Let M_1, M_2, M_3 be finite sets of primes, whose union consists of different primes. If the prime divisors of the natural number Z_1 belong to M_1 , the prime divisors of the natural number Z_2 to M_2 and the prime divisors of the natural number Z_3 to M_3 , then the equation

$$Z_1 + Z_2 = Z_3$$

has only a finite number of solutions.

In his proof he sets $M_1 = \{P_1, P_2, \dots, P_t\}$, P_i prime, $1 \leq i \leq t$, $M_2 = \{Q_1, Q_2, \dots, Q_u\}$, Q_i prime, $1 \leq i \leq u$, $M_3 = \{R_1, R_2, \dots, R_v\}$, R_i prime, $1 \leq i \leq v$, so that the equation $Z_1 + Z_2 = Z_3$ becomes

$$P_1^{p_1} P_2^{p_2} \dots P_t^{p_t} + Q_1^{q_1} Q_2^{q_2} \dots Q_u^{q_u} = R_1^{r_1} R_2^{r_2} \dots R_v^{r_v},$$

where the exponents $p_1, p_2, \dots, p_t, q_1, q_2, \dots, q_u, r_1, r_2, \dots, r_v$ are non-negative integers.

It is this form of equation we shall use in our application of Mahler's theorem.

4. The finiteness of the classes $C_{p_1 \dots p_\omega}$

Theorem. For a given fixed radical $p_1 \dots p_\omega$ the class $C_{p_1 \dots p_\omega}$ contains only a finite number of abc-equations.

Proof. Suppose the opposite is true. Then there is an infinite number of abc-equations $c = a + b$ with radical $R(abc) = p_1 \dots p_\omega$.

Since, according to §2, for a given fixed ω there are only finitely many types, there must be at least one type $T(\underbrace{p \dots p}_\kappa, \underbrace{p \dots p}_\lambda, \underbrace{p \dots p}_\mu)$, $\kappa + \lambda + \mu = \omega$ for

which there is an infinite number of abc-equations with $R(abc) = p_1 \dots p_\omega$. Let these equations be

$$q_1^{x_{i1}} \cdots q_\kappa^{x_{i\kappa}} = r_1^{y_{i1}} \cdots r_\lambda^{y_{i\lambda}} + s_1^{z_{i1}} \cdots s_\mu^{z_{i\mu}}$$

$$x_{ij} \geq 1, \quad 1 \leq j \leq \kappa$$

$$y_{ij} \geq 1, \quad 1 \leq j \leq \lambda$$

$$z_{ij} \geq 1, \quad 1 \leq j \leq \mu$$

$$i = 1, 2, \dots, \infty$$

where $\{q_1, \dots, q_\kappa\}$, $\{r_1, \dots, r_\lambda\}$, $\{s_1, \dots, s_\mu\}$ are disjoint sets of primes satisfying

$$\{q_1, \dots, q_\kappa\} \cup \{r_1, \dots, r_\lambda\} \cup \{s_1, \dots, s_\mu\} = \{p_1, \dots, p_\omega\}.$$

Above equations with their respective constraints are exactly those considered by Mahler, excluding the equations where at least one of the exponents $p_1, p_2, \dots, p_t, q_1, q_2, \dots, q_u, r_1, r_2, \dots, r_v$ is zero. Applying his theorem it follows that the supposition that there are infinite such equations leads to a contradiction.

Hence the class $C_{p_1 \dots p_\omega}$ contains only a finite number of abc-equations. \square

Note. Above result is an immediate consequence of the abc-conjecture, assumed to be true.

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References

- [1] Kurt Mahler, Zur Approximation algebraischer Zahlen. I. (Über den grössten Pronteiler binärer Formen), Math. Ann. 107(1933),p.691-730.
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